Stochastic Hybrid Processes as Solutions of Stochastic Differential Equations

Estimation of Rare Event Probability in Stochastic Hybrid Systems

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Outline

- Introduction
- Classical SDE models
- Stochastic Hybrid Systems
  - Model by Ghosh and Bagchi
  - Models by Krystul and Blom
- Estimation of rare event probability in stochastic hybrid systems

Stochastic Differential Equation

$$dX_t = a(X_t)\,dt + \sigma(X_t)\,dW_t, \quad t > 0$$

$$X(0) = X_0$$

$\{W_t, t \geq 0\}$ is the standard Brownian motion.
$a(\cdot)$: drift coefficient and $b(\cdot)$: diffusion coefficient

Solution : existence and uniqueness; under "Lipschitz" and "Growth" condition

(L) $\quad |a(x) - a(y)|^2 + |b(x) - b(y)|^2 \leq K|x - y|^2$

(G) $\quad |a(x)|^2 + |b(x)|^2 \leq K|x|^2$

SDE: Strong solution: Properties

1. Continuous path

2. Semimartingale : $X(t) = X_0 + A(t) + M(t)$

   $A(t)$ is of bounded variation, $M(t)$ is a Martingale.

   – Important for mathematical analysis (well developed theory)

3. Markov : $\mathcal{D}\left(X(t)|\mathcal{F}_s\right) = \mathcal{D}\left(X(t)|X(s)\right), \quad t > s.$

   Transition prob. : $p(s, x, t, A) = P(X(t) \in A | X(s) = x)$

   Time homogeneous : $p(s, x, t, A) = p(0, x, t-s, A)$

   $$\Rightarrow \mathcal{D}\left(X(t+s)|\mathcal{F}_s\right) = \mathcal{D}_{X(s)}\left(X(t)\right).$$

   Strong Markov :

   $\mathcal{D}\left(X(t+\tau)|\mathcal{F}_\tau\right) = \mathcal{D}_{X(\tau)}\left(X(t)\right), \ \tau$ stopping time

   – Helpful to analyse processes stopped at random times

Jump Diffusion Process

- Extension while keeping Semimartingale property?
- Discontinuous paths?

Jump Diffusion Process :

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t + h(X_t)\,dN_t, \quad t > 0$$

$\{N(t)\}$: (pure) jump process.

Alternative representation:

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t + \int_{\mathbb{R}} g(X_{t-}, u)\,p(dt, du)$$

$p(\cdot, \cdot)$ Poisson random measure associated with the jump process.

Solution: existence, uniqueness under Growth and Lipschitz condition on a, b and g.

Properties : Semimartingale, Markov, but *Discontinuous*

Stochastic Hybrid Systems

Further extension : Not only the process but also the governing model/equation jumps.

— *regime switch* or *Hybrid* model.

Model I (Ghosh & Bagchi):

$$(X_t, \theta_t) \in \mathbb{R}^d \times \mathbb{M}, \ \mathbb{M} = \{1, \ldots, N\}, \ t \geq 0$$

$$dX_t = a(X_t, \theta_t)\, dt + b(X_t, \theta_t)\, dW_t + \int_{\mathbb{R}} g(X_{t-}, \theta_{t-}, u)\, p(dt, du)$$

$$P\left(\theta_{t+\delta t} = j | \theta_t = i, X_s, \theta_s, s \leq t\right) = \lambda_{ij}(X_t)\delta t + o(\delta t), \quad i \neq j$$

$$X(0) = X_0 \quad \theta(0) = \theta_0$$

$p(\cdot, \cdot)$ — Poisson random measure with intensity $dt \times l(du)$

$\lambda_{ij}(\cdot) \geq 0, \ \ i \neq j, \ i, j = 1, 2, \ldots N$ and $\sum_{j=1}^{N} \lambda_{ij}(\cdot) = 0$

Hybrid Systems - Model 1

Assumptions :

- $a(\cdot, i),\ b(\cdot, i)$ are bounded and Lipschitz continuous
- $\lambda_{ij}(\cdot)$ are bounded and measurable
- Support of $g(x, \theta, u)$ w.r.t. 'u' (i.e., the proj. on $\mathbb{R}$) is bdd.

Existence of Solution: (Put in Ito-Skorohod's framework)

Identify i with e_i, ith unit vector in $\mathbb{R}^N$ and embed $\mathbb{M}$ into $\mathbb{R}^N$. For $x \in \mathbb{R}^d$, define $\Delta_{ij}(x)$ to be the consecutive (with lexicographic ordering on $(i, j) \in \mathbb{M} \times \mathbb{M}$) left closed right open intervals on $\mathbb{R}_+$ with length $\lambda_{ij}(x)$.

Define $c : \mathbb{R}^d \times \mathbb{M} \times \mathbb{R} \to \mathbb{R}^N$ as

$$c(x, i, u) = e_j - e_i \quad \text{if} \quad u \in \Delta_{ij}(x), \qquad 0 \text{ otherwise.}$$

Hybrid Systems - Model 1

Model can be expressed as

$$dX_t = a(X_t, \theta_t)\, dt + b(X_t, \theta_t)\, dW_t + \int_{\mathbb{R}} g(X_{t-}, \theta_{t-}, u)\, p(dt, du)$$

$$d\theta_t = \int_{\mathbb{R}} c(X_{t-}, \theta_{t-}, u)\, p(dt, du), \quad t > 0$$

$$X(0) = X_0 \quad \theta(0) = \theta_0$$

and the existence of unique strong solution can be shown.

Remarks :

- Conditions on the coefficients can be relaxed
- $\{X(t)\}$ is not Markov
- Augmented process $(X(t), \theta(t))$ is Markov

Hybrid Systems - Model 1

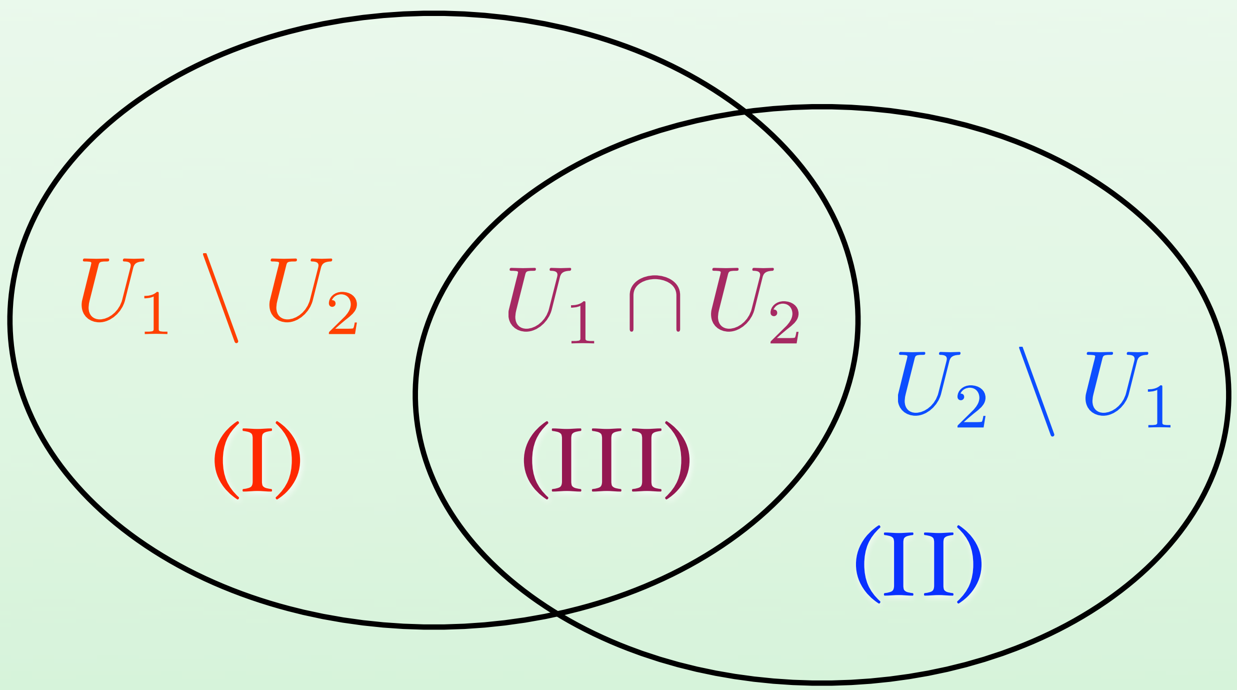
Let U_1 and U_2 denote respectively the supports of $g(x, \theta, u)$ and $c(x, \theta, u)$ w.r.t. 'u' (i.e. the projections on $\mathbb{R}$).

Different situations can arise:

I. Process jumps but no switching: $u \in U_1 \setminus U_2$

II. No jump in process but switching occurs: $u \in U_2 \setminus U_1$

III. Simultaneous switching and jump (Hybrid Jump): $u \in U_1 \cap U_2$



Hybrid Systems - Model 2 (Krystul and Blom)

$$dX_t = a(X_t, \theta_t)\, dt + b(X_t, \theta_t)\, dW_t + \int_{\mathbb{R}^n} g_1(X_{t-}, \theta_{t-}, u)\, q_1(dt, du)$$
$$+ \int_{\mathbb{R}^n} g_2(X_{t-}, \theta_{t-}, u)\, p_2(dt, du)$$

$$d\theta_t = \int_{\mathbb{R}^n} c(X_{t-}, \theta_{t-}, u)\, p_2(dt, du), \quad t > 0$$

- W is an m-dimensional standard Wiener process.
- $q_1(dt, du)$ is a martingale random measure associated to a Poisson random measure p_1 with intensity $dt \times m_1(du)$.
- $p_2(dt, du)$ is a Poisson random measure with intensity $dt \times m_2(du) = dt \times du_1 \times \mu(\underline{u})$, μ is a probability measure on $\mathbb{R}^{n-1}$, $u_1 \in \mathbb{R}$.

Hybrid Systems - Model 2

Assumptions :

- Lipschitz condition. For all $i = 1, 2, \dots, N$

$$|a(x, e_i)^2| + |b(x, e_i)|^2 + \int_{\mathbb{R}^n} |g_1(x, e_i, u)|^2 m_1(du) \le K(1 + |x|^2).$$

- Growth condition:

$$|a(x, e_i) - a(y, e_i)|^2 + |b(x, e_i) - b(x, e_i)|^2$$
$$+ \int_{\mathbb{R}^n} |g_1(x, e_i, u) - g_1(x, e_i, u)|^2 m_1(du) \le K_r |x - y|^2.$$

for $|x| \le r$, $|y| \le r$.

- $\lambda_{ij}(\cdot)$ are bounded and measurable
- Support of $g_2(x, \theta, u)$ w.r.t. 'u_1' (i.e., the proj. on $\mathbb{R}$) is bdd.

Hybrid Systems - Model 3 (Krystul, Blom and Bagchi)

Idea:

X_t starts in some set E^i and (X_t, θ_t) evolves according to some Ito-Skorohod SDE.

If X_t reaches a boundary of that set, the process immediately jumps to a new set E^j (possibly including jump in θ)

Process continues again according to the Ito-Skorohod SDE in a new set until it reaches the boundary of the current set.

$$E = \{x \,|\, x \in E^i, \text{ for some } i = 1, \ldots, L\} = \bigcup_{i=1}^{L} E^i$$

$$\partial E = \{x \,|\, x \in \partial E^i, \text{ for some } i = 1, \ldots, L\} = \bigcup_{i=1}^{L} \partial E^i$$

Hybrid Systems - Model 3

$(X_t, \theta_t) \in \mathbb{R}^d \times \mathbb{M}, \quad \mathbb{M} = \{e_1, e_2, \dots, e_N\}.$

Suppose $\tau_1^E < \tau_2^E < \cdots < \tau_m^E < \dots.$ are the successive hitting times of ∂E.

Between the boundary jumps $\{X_t, \theta_t\}$ is a switching jump-diffusion process. For $\tau_m^E < t < \tau_{m+1}^E$,

$$X_t = X_{\tau_m^E} + \int_{\tau_m^E}^t a(X_s, \theta_s) ds + \int_{\tau_m^E}^t b(X_s, \theta_s) dW_s + \int_{\tau_m^E}^t \int_{\mathbb{R}^n} g_2(X_{s-}, \theta_{s-}, u) p_2(ds, du),$$

$$\theta_t = \theta_{\tau_m^E} + \int_{\tau_m^E}^t \int_{\mathbb{R}^n} c(X_{s-}, \theta_{s-}, u) p_2(ds, du).$$

Hybrid Systems - Model 3

At jump points ($t = \tau_m$),

$$
\begin{aligned}
X_{\tau_m^E} &= f^x(X_{\tau_m^E -}, \theta_{\tau_m^E -}, Z), \\
\theta_{\tau_m^E} &= f^\theta(X_{\tau_m^E -}, \theta_{\tau_m^E -}, Z),
\end{aligned}
$$

where Z is some V-valued random variable,

$$
\begin{aligned}
f^x &: \partial E \times \mathbb{M} \times V \longrightarrow E, \\
f^\theta &: \partial E \times \mathbb{M} \times V \longrightarrow \mathbb{M}.
\end{aligned}
$$

Hybrid Systems - Model 3

Assumptions :

- a and b satisfy Lipschitz and Growth conditions
- $\lambda_{ij}(\cdot)$ are bounded and measurable
- Support of $g_2(x, \theta, u)$ w.r.t. 'u_1' (i.e., the proj. on $\mathbb{R}$) is bdd.
- Function g_2 has the following property: $(x + g_2(x, \theta, u)) \in E$ for each $x \in E^i, \theta \in \mathbb{M}, u \in \mathbb{R}^n, i = 1, \ldots, L$
- $d(\partial E, f^x(\partial E, \mathbb{M}, V)) > 0,$

  i.e. when $\{X_t\}$ has reached the boundary ∂E it always jumps inside of open set E.

Estimation of rare event probability in stochastic hybrid systems

$\{X_t, \theta_t\} \in \mathbb{R}^n \times \mathbb{M}, \quad \mathbb{M} = \{e_1, \ldots, e_S\}$

switching diffusion process:

$$dX_t = a(\theta_t, X_t)dt + b(\theta_t, X_t)dW_t,$$
$$P(\theta_{t+\delta} = \theta | \theta_t = \eta, X_t = x) = \lambda_{\eta\theta}(x)\delta + o(\delta), \eta \neq \theta.$$

(X_t, θ_t) starts at $t = 0$ in $D_0 \subset \mathbb{R}^n \times \mathbb{M}$ with known initial probability distribution $P_{X_0,\theta_0}(\cdot)$.

$\tau_A = \inf\{t \geq 0 : X_t \in A\}$: hitting time of $A \subset \mathbb{R}^n, \ A \cap D_0 = \varnothing$.

To calculate $P(\tau_A < T)$: probability that X_t will reach the set A in the time interval $(0, T)$.

Using crude Monte Carlo to estimate $P(\tau_A < T)$ may require a huge number of simulations. For events with probability of the order 10^{-10} , often 10^{11} or more Monte Carlo simulated trajectories are needed.

Factorization Approach

Idea : Identify intermediate states that are (sequentially) visited much more often than the rare target set.

D_0 initial set;

$D_1 \supset \cdots \supset D_{m-1} \supset D_m = A \subset \mathbb{R}^n; \quad D_0 \cap D_1 = \varnothing.$

Define for $k = 1, \ldots, m$, stopping times $\tau_k \triangleq \inf\{t \geq 0 : X_t \in D_k\}$

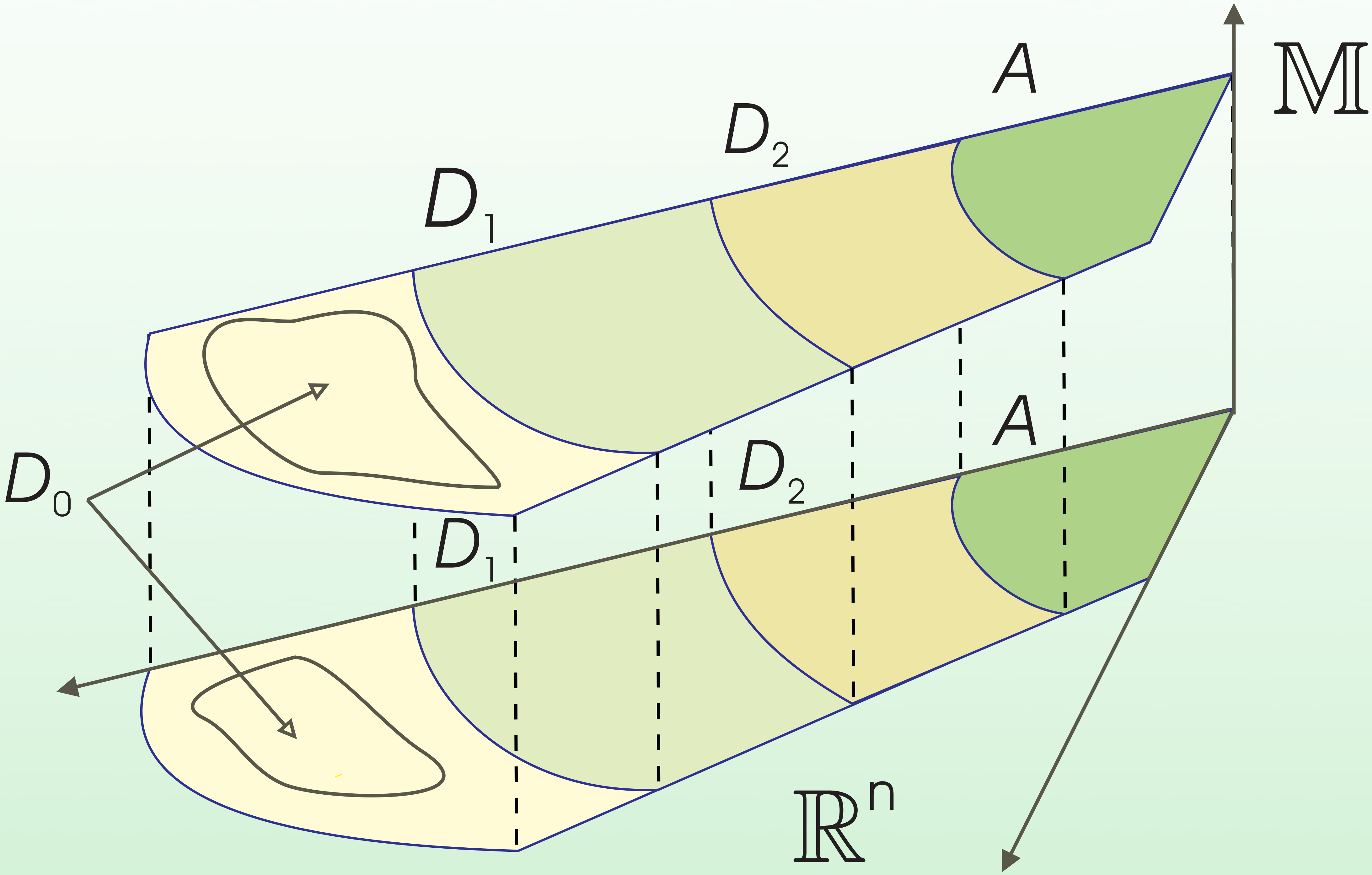
$$
\begin{aligned}
P(\tau_A < T) &= P(\tau_m < T) \\
&= P(\tau_m < T, \tau_{m-1} < T, \ldots, \tau_0 < T) \\
&= \prod_{k=1}^{m} P(\tau_k < T | \tau_{k-1} < T).
\end{aligned}
$$

Individual probabilities on r.h.s. are not very small.

Nested sequence of level sets

D_0 **initial set;** $D_1 \supset \cdots \supset D_{m-1} \supset D_m = A;\quad D_0 \cap D_1 = \varnothing.$

$P(\tau_A < T) = \prod_{k=1}^{m} P(\tau_k < T | \tau_{k-1} < T).$



Factorization Approach (continued)

$$P(\tau_A < T) = \prod_{k=1}^{m} P(\tau_k < T | \tau_{k-1} < T) \triangleq \boxed{\prod_{k=1}^{m} \gamma_k}$$

To calculate γ_k's we proceed as follows:

Define $\xi_k = (X_{\tau_k \wedge T}, \theta_{\tau_k \wedge T})$,

$$\pi_k(B) = P(\xi_k \in B | \tau_0 < T, \ldots, \tau_k < T), \qquad B \subset \mathbb{R}^n \times \mathbb{M},$$

$$p_k(B) = P(\xi_k \in B | \tau_0 < T, \ldots, \tau_{k-1} < T)$$

$\{(X_t, \theta_t)\}$ strong Markov $\Rightarrow$ $\{\xi_k\}$ is a Markov (chain)

Factorization Approach (continued)

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$\{(X_t, \theta_t)\}$ strong Markov $\Rightarrow$ $\{\xi_k\}$ is a Markov (chain)

$$p_k(B) = \int_{\mathbb{R}^n \times \mathbb{M}} P_{\xi_k | \xi_{k-1}}(B|\xi) \pi_{k-1}(d\xi),$$

$$\pi_k(B) = \frac{P(\xi_k \in B, \tau_0 < T, \ldots, \tau_k < T)}{P(\tau_0 < T, \ldots, \tau_k < T)} = \frac{P(\xi_k \in B \cap \bar{D}_k, \tau_0 < T, \ldots, \tau_{k-1} < T)}{P(\tau_0 < T, \ldots, \tau_k < T)}$$

$$= \frac{P(\xi_k \in B \cap \bar{D}_k | \tau_0 < T, \ldots, \tau_{k-1} < T)}{\boxed{P(\tau_k < T | \tau_0 < T, \ldots, \tau_{k-1} < T)}} = \frac{\int_B \mathbf{I}_{\{\xi \in \bar{D}_k\}} p_k(d\xi)}{\int_{\mathbb{R}^n \times \mathbb{M}} \mathbf{I}_{\{\xi \in \bar{D}_k\}} p_k(d\xi)}$$

$$\bar{D}_k \triangleq D_k \times \mathbb{M}$$

$$\boxed{\gamma_k = \int_{\mathbb{R}^n \times \mathbb{M}} \mathbf{I}_{\{\xi \in \bar{D}_k\}} p_k(d\xi)}$$

Factorization Approach (continued)

$$P(\tau_A < T) = \prod_{k=1}^{m} P(\tau_k < T | \tau_{k-1} < T) \triangleq \boxed{\prod_{k=1}^{m} \gamma_k}$$

To calculate γ_k's we proceed as follows:

Define
$$\xi_k = (X_{\tau_k \wedge T}, \theta_{\tau_k \wedge T}),$$
$$\pi_k(B) = P(\xi_k \in B | \tau_0 < T, \ldots, \tau_k < T), \qquad B \subset \mathbb{R}^n \times \mathbb{M},$$
$$p_k(B) = P(\xi_k \in B | \tau_0 < T, \ldots, \tau_{k-1} < T)$$

$\{(X_t, \theta_t)\}$ strong Markov $\Rightarrow$ $\{\xi_k\}$ is a Markov (chain)

Evolution of the flow $\{\pi_k, p_k, \gamma_k; k = 0, 1, \ldots, m\}$ is described by the following diagram:

$$\begin{tikzcd} \pi_{k-1} \arrow[r, "\text{prediction}"] & p_k \arrow[r, "\text{update}"] \arrow[d] & \pi_k \\ & \gamma_k & \end{tikzcd}$$

with initial condition $\pi_0(d\xi) = P_{\xi_0}(d\xi) = P(\xi_0 \in d\xi)$.

Interacting Particle System (IPS) approach

An approximating sequence $\{\hat{\pi}_k, \hat{p}_k, \hat{\gamma}_k; k = 0, 1, \ldots, m\}$:

$$
\begin{array}{ccccc}
\hat{\pi}_{k-1} & \xrightarrow{\text{prediction}} & \hat{p}_k & \xrightarrow{\text{update}} & \hat{\pi}_k \\
 & & \downarrow & & \\
 & & \hat{\gamma}_k & &
\end{array}
$$

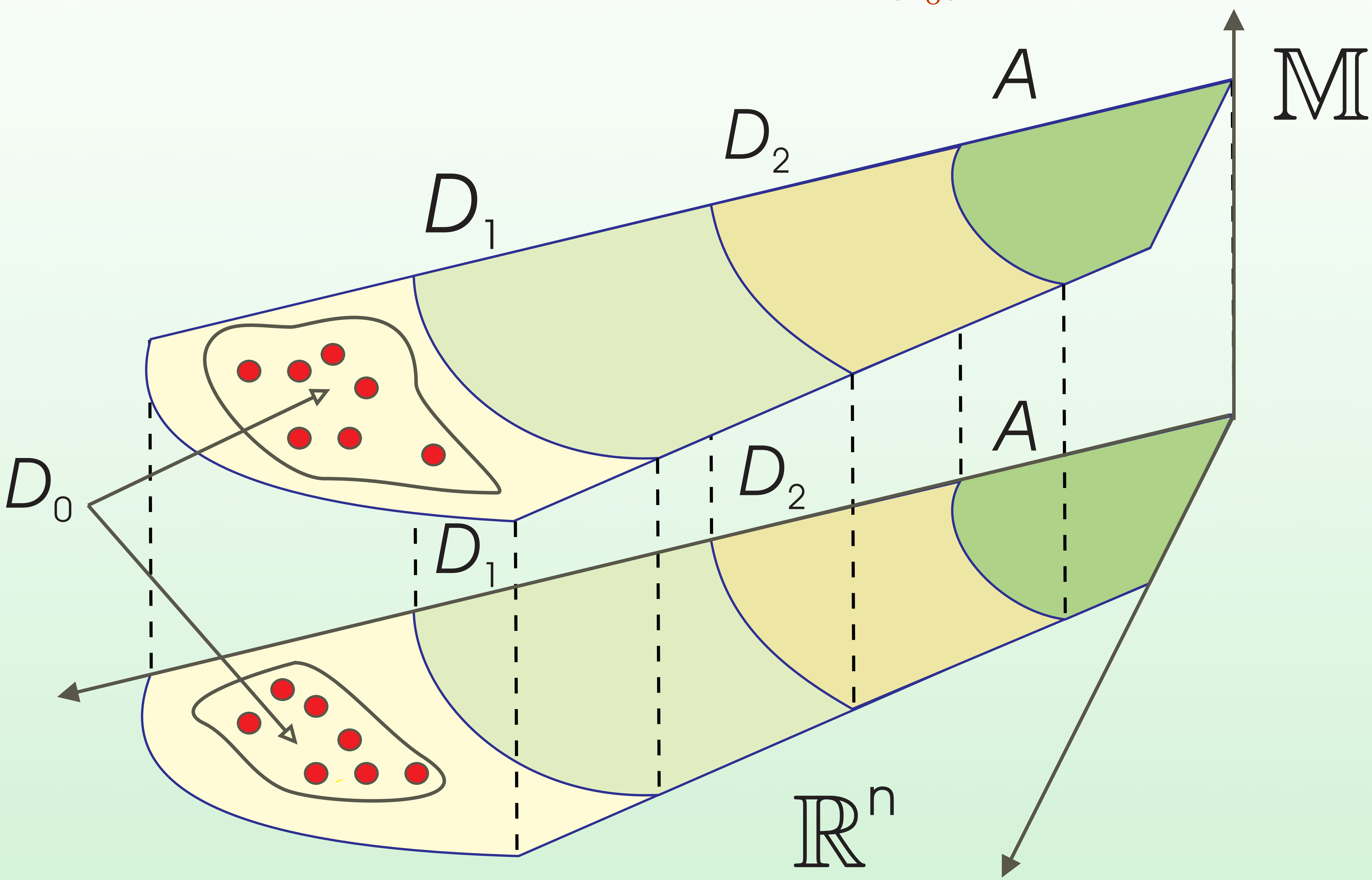
Approximations in the form of weighted empirical distributions associated with the particle system $\{\xi_k^i, \omega_k^i\}_{i=1}^N$:

$$
\pi_0 \approx \hat{\pi}_0 = \sum_{i=1}^N \omega_0^i \delta_{\{\xi_0^i\}}, \qquad \pi_k \approx \hat{\pi}_k = \sum_{i=1}^N \frac{\mathbf{I}_{\{\xi_k^i \in \bar{D}_k\}}}{\sum_{j=1}^N \omega_k^j \mathbf{I}_{\{\xi_k^j \in \bar{D}_k\}}} \delta_{\{\xi_k^i\}},
$$

$$
p_k \approx \hat{p}_k = \sum_{i=1}^N \omega_k^i \delta_{\{\xi_k^i\}}, \qquad \gamma_k \approx \hat{\gamma}_k = \sum_{i=1}^N \omega_k^i \mathbf{I}_{\{\xi_k^i \in \bar{D}_k\}}.
$$

IPS algorithm

Step 1 : At $t = 0$ choose N particles $\{\xi_0^i, i = 1, \ldots N\}$ from $P_{X_0,\theta_0} \equiv \pi_0$, equivalently, $\hat{\pi}_0 \triangleq \sum_{i=1}^{N} w_0^i \delta_{\{\xi_0^i\}}$, $w_0^i = \frac{1}{N}$.



IPS algorithm (continued)

Step 2. Prediction step: $\pi_{k-1} \longrightarrow p_k$

Given $\hat{\pi}_{k-1} \triangleq \sum_{i=1}^{N} w_{k-1}^i \delta_{\{\xi_{k-1}^i\}}$, i.e., a weighted particle cloud $\{(\xi_{k-1}^i, w_{k-1}^i), i = 1, \dots N\}$ with $\xi_{k-1}^i \in \bar{D}_{k-1}$,

Let particles evolve following the hybrid system equations until $\bar{D}_k$ or the final time T is hit.

$\hat{\xi}_k^i$: Value of the i-th particle at the end of the step

$\hat{p}_k = \sum_{i=1}^{N} w_{k-1}^i \delta_{\{\hat{\xi}_k^i\}}$ is the approximation of p_k.

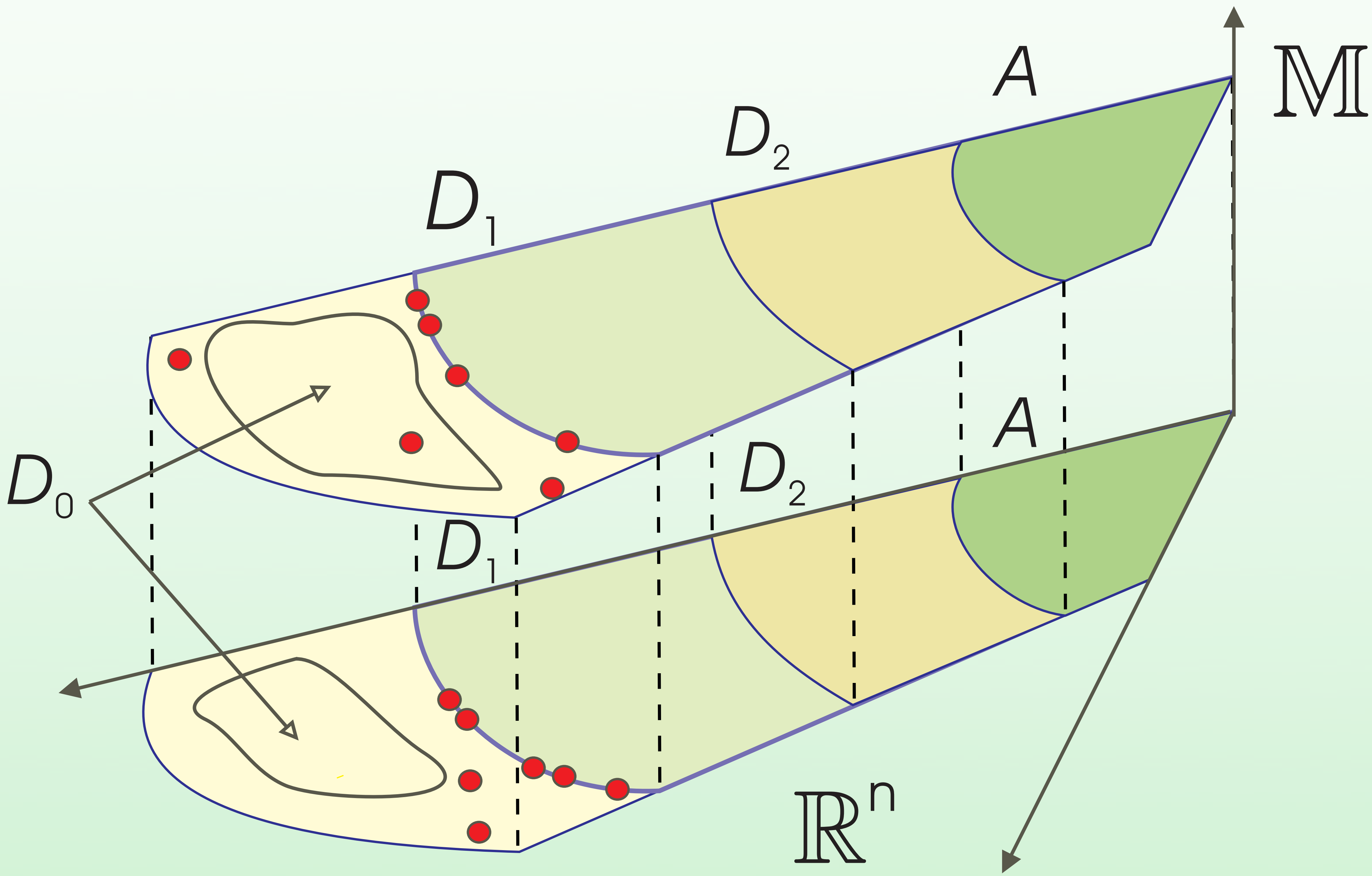
Recall that:

$p_k(B) = P(\xi_k \in B | \tau_0 < T, \dots, \tau_{k-1} < T)$, $B \subset \mathbb{R}^n \times \mathbb{M}$.

IPS algorithm (continued)

$\hat{\xi}_k^i$: Value of the i-th particle at the end of the prediction step

$\hat{p}_k = \sum_{i=1}^N w_{k-1}^i \delta_{\{\hat{\xi}_k^i\}}$ is the approximation of p_k.



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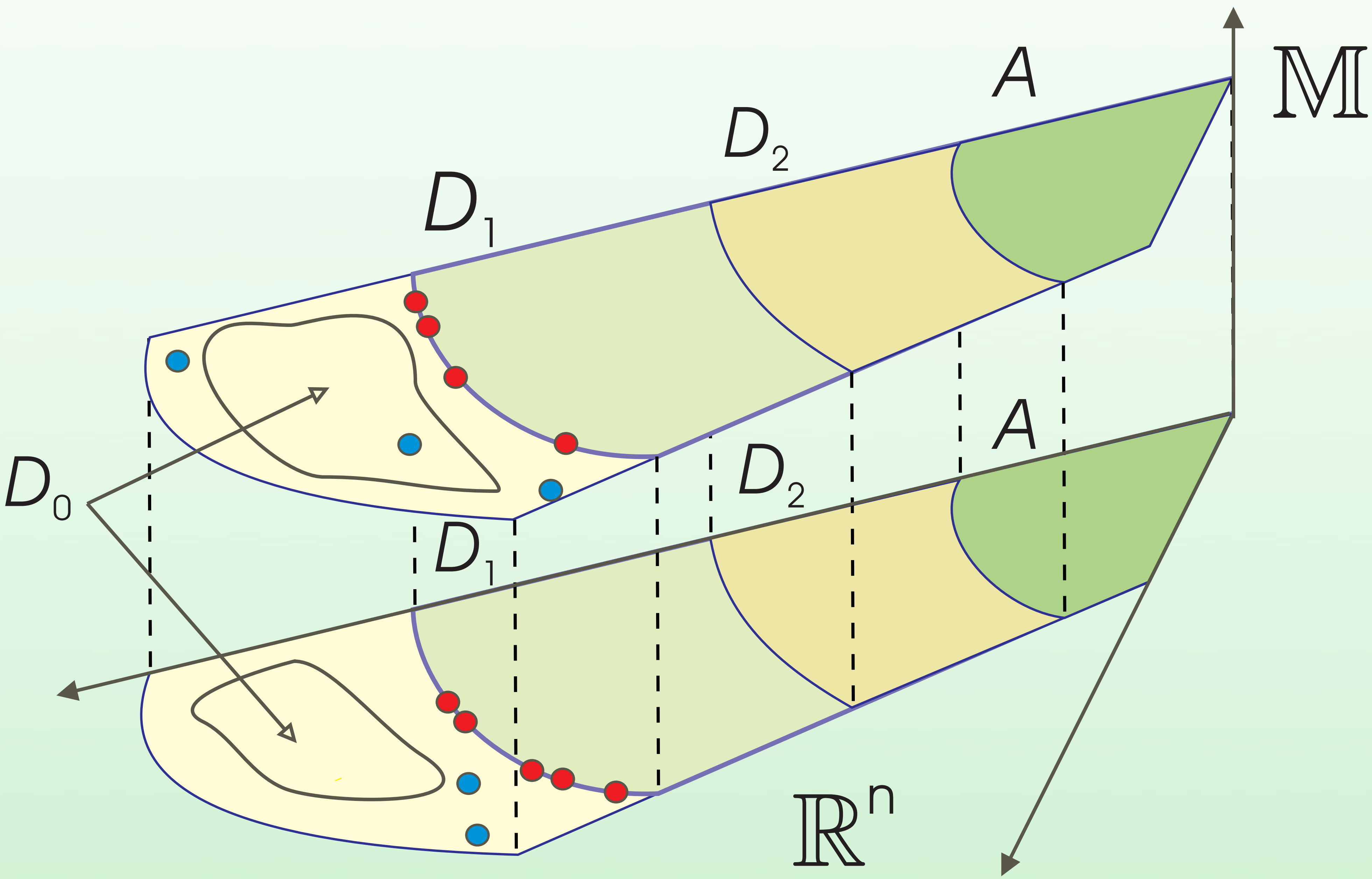
IPS algorithm (continued)

Set $\hat{w}_k^i = 0$, if $X^i_{\tau_k \wedge T} \notin D_k$, otherwise $\hat{w}_k^i = w_{k-1}^i$.

Approximate $\gamma_k = P(\tau_k < T | \tau_{k-1} < T)$ by $\hat{\gamma}_k = \sum_{i=1}^N \hat{w}_k^i$.

Stop algorithm if $\hat{\gamma}_k = 0$, and $P(\tau_A < T) \approx \prod_{k=1}^m \hat{\gamma}_k = 0$.



IPS algorithm (continued)

Step 3. Updating step: $p_k \longrightarrow \pi_k$

Approximate π_k by $\hat{\pi}_k \triangleq \sum_{i=1}^{N} \tilde{w}_k^i \delta_{\{\hat{\xi}_k^i\}}$, with $\tilde{w}_k^i = \hat{w}_k^i \Big/ \sum_{i=1}^{N} \hat{w}_k^i$.

Recall that: $\pi_k(B) = P(\xi_k \in B | \tau_0 < T, \ldots, \tau_k < T)$, $B \subset \mathbb{R}^n \times \mathbb{M}$.

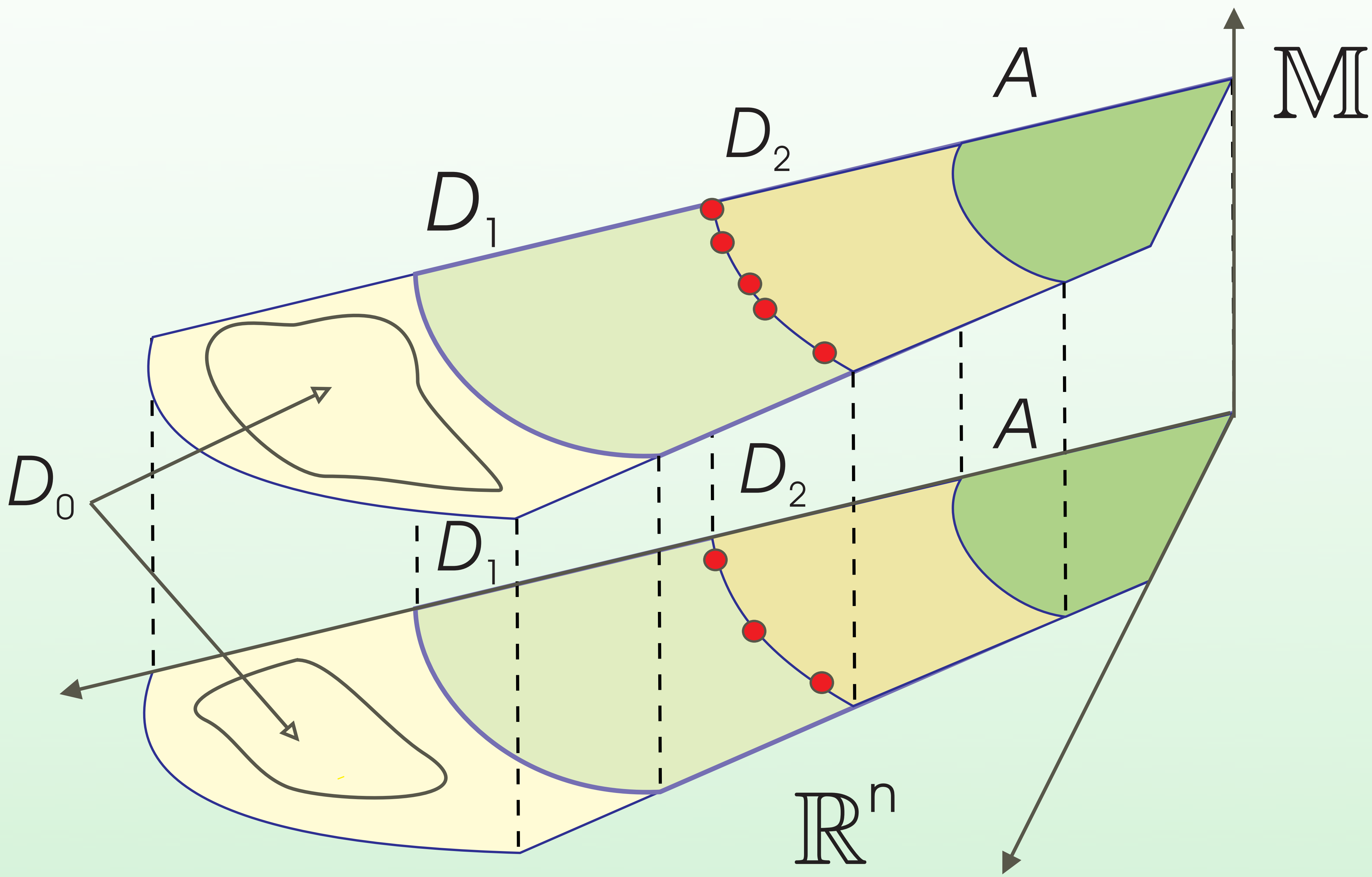
To avoid carrying the particles with no weight, resample N particles from $\hat{\pi}_k$.

The new set of particles is $\{\xi_k^i, \omega_k^i\}_{i=1}^{N}$ with $w_k^i = \frac{1}{N}$.

Go to Step 2.

IPS algorithm (continued)

$k := k + 1$



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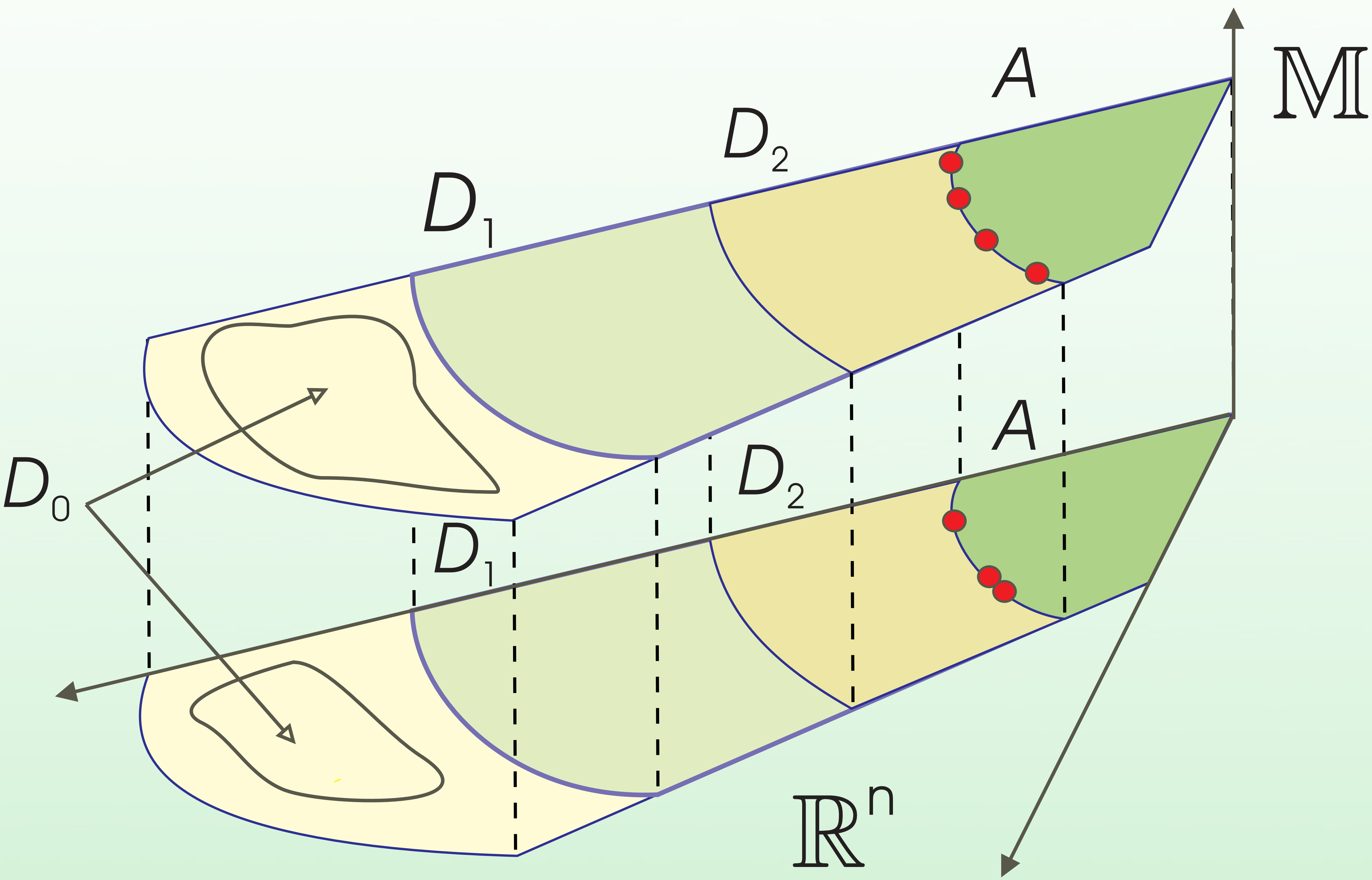
IPS algorithm (continued)

At stage $k = m$ we get $P(\tau_A < T) \approx \prod_{k=1}^{m} \hat{\gamma}_k$.



Theorem (Cerou, Del Moral, LeGland and Lezaud, 2002)

IPS estimator is unbiased, i.e.

$$\mathbb{E}\Big[\prod_{k=1}^{m}\gamma_k^N\Big] = P(\tau_A < T) = P_{hit}(0,T)$$

and

$$\left(\mathbb{E}\Big(\prod_{k=1}^{m}\gamma_k^N - \prod_{k=1}^{m}\gamma_k\Big)^p\right)^{\frac{1}{p}} \leq \frac{a_p b_m}{\sqrt{N}},$$

for some finite constant a_p which depends only on the parameter p, and for some finite constant b_m which depends only on the parameter m.

Diffusion Example: Geometric Brownian Motion

$$dX_t = (\mu + \tfrac{\sigma^2}{2})X_t dt + \sigma X_t dW_t, \quad X_0 = x$$

Probability of hitting level d before time T :

$P(\tau_d \leq T), \quad \tau_d \triangleq \inf\{t > 0 : X_t \geq d\}.$

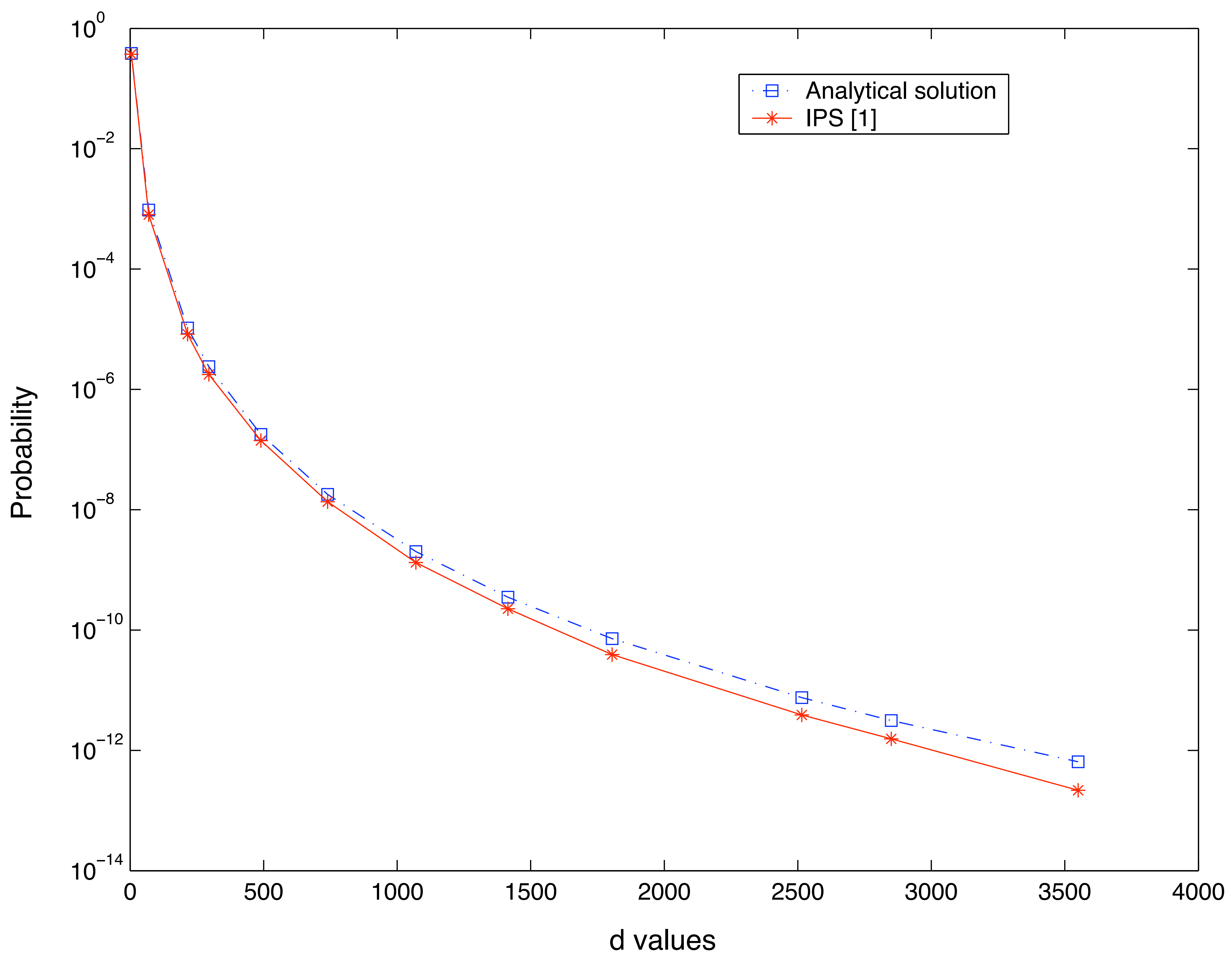
Analytical solution: $\displaystyle\int_0^t \frac{\ln(d/x)}{\sqrt{2\pi\sigma^3 s^3}} \exp\{\frac{-(\ln(d/x) - \mu s)^2}{2\sigma^2 s}\} ds$

IPS :

- $\mu = 1, \sigma = 1, x_0 = 1,$ varying $d \leq 3550$
- intermediate levels d_j 's are chosen experimentally; 40%–55% of particles starting at D_{j-1} reaches D_j.
- 1000 simulations of 1000 particles each

$$P_{hit}(0, T) \approx \tfrac{1}{1000} \textstyle\sum_{i=1}^{1000} (\prod_{k=1}^{m} \hat{\gamma}_k)^{(i)}$$

Probability to hit level d before time $T = 1$: diffusion



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